For arbitrary positive values of ωt and ϕ,

$$\frac{\Delta x}{\Delta z} = \left(\frac{\Delta x}{\Delta z}\right)_{\phi=0} \cos\phi + \left[\frac{\omega^2 R/g}{(\omega t/2) + (\omega U_z/g)}\right] \times \left\{\left(\frac{\sin\omega t}{\omega t} - 1\right)\sin\phi + \left(\frac{4\sin^2(\omega t/2)}{\omega^2 t^2} - 1\right)\frac{\omega t}{2}\cos\phi\right\}$$

is obtained. If $|\phi| \sim |\omega t + \phi| < \frac{1}{3}$ rad, the term in the braces is less than 10^{-2}, and the correction term to $(\Delta x/\Delta z)_0$ is less than 5%, a relatively insignificant correction.

Hence, the kinematic description of the motion is consistent with observations.

The radial acceleration A_r with which a drop falls away from the rotating surface is given as $A_r = (\Delta x/\Delta z)(g/\omega^2 R) \times \omega^2 R$. When $\Delta x/\Delta z$ and $g/\omega^2 R$ are taken from the test values, $A_r \cong 0.8\, \omega^2 R$ is found.

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Liquid Sloshing in a Cylindrical Quarter Tank

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With the increasing size of space vehicles and their larger diameter, both of which lower the natural frequencies of the propellants, the effects of propellant sloshing upon the stability of the vehicle are becoming more critical, especially since at launch usually a very large amount of the total mass is in the form of liquid propellant. With increasing diameter, the oscillating propellant masses and the corresponding forces increase. Furthermore, the natural frequencies of the propellant become smaller and shift closer to the control frequency of the space vehicle. A relatively simple means of avoiding strong dynamic coupling of the propellant motion and the control system is represented by compartmentation of propellant containers with longitudinal walls. This results in smaller sloshing masses and larger natural frequencies. Free and forced liquid oscillations in form of translatory, pitching, and roll excitation have been determined for a cylindrical container of circular quarter cross section. The fluid was assumed to be irrotational, inviscid, and incompressible. The velocity potential of the liquid is obtained from the solution of Laplace's equation with linearized boundary conditions. Forces and moments of the liquid are obtained by integration of the pressure distribution along the container walls. The results of the theoretical studies compared with available experimental values are in good agreement.

I. Introduction

IN space boosters, the diameter of the propellant tanks becomes rather large, and the response of the vehicle to the motions of the container liquid will greatly affect the stability and control of the space vehicle. Propellant oscillations are important because there is a possibility of extreme amplitudes if the excitation frequency is in the neighborhood of one of the natural modes of the propellant. Since a very large amount of the total weight of the vehicle is in the form of liquid propellant, the influence of propellant sloshing upon the stability of the vehicle becomes more critical with increasing tank diameter. The close grouping of control frequency and natural frequencies of the propellant, the relatively low structural frequencies, the very rapid increase of the oscillating propellant masses, and propellant forces with increasing diameter demand thorough investigation of this phenomenon.

The problem of free fluid oscillations in a circular cylindrical container was treated in 1829 by Poisson. Because the theory of Bessel functions was unavailable at the time, the result was not completely interpreted.[1] In 1876, Rayleigh[2] gave the solution for free oscillations in rectangular and cylindrical tanks of circular cross section. In recent years, the problem of forced fluid oscillations has grown in importance.[3] Graham and Rodrigues[4] determined the forced vibration of liquid in rectangular containers, whereas[5] Lorell gave the flow of a fluid in a two-dimensional rectangular container and cylindrical tank of circular cross section for translational excitation. Almost at the same time, many reports appeared about forced fluid oscillations in cylindrical tanks.[6–9] Fluid oscillations in cylindrical tanks with annular[10] and elliptic[11] cross section as well as those in horizontal circular cylindrical tanks and spherical containers[12] have also been treated.

The natural frequencies of the propellant in cylindrical tanks with circular cross section is proportional to the inverse of the root of the tank diameter, whereas the sloshing masses exhibit rather large magnitudes.

To eliminate the unfavorable effect of the propellant motion upon the stability of a space vehicle, various measures can be explored. Internal damping in the liquid can be introduced by fixed baffles[13] or movable slosh suppression devices such as rigid lids following the free propellant surface or floating bodies partially submerged below the free fluid surface. However, use of moving parts is usually avoided because of structural and weight reasons.

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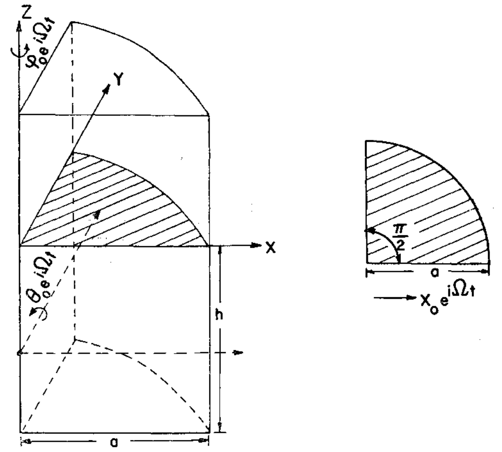

Fig. 1 Tank geometry and coordinate system.

Annular ring baffles introduce damping that is proportional to the square root of the surface displacement at the tank wall and to the three-halves power of the effective ring area. Furthermore, the location of the baffle has a strong influence and is approximately proportional to the reciprocal e-function. These annular ring baffles, which are attached at the inner wall of the tank, reduce the peak forces and moments of the propellant. However, the main influence, which comes from the sloshing mass, is not considerably improved. Another possible remedy is to subdivide the tanks with longitudinal walls, such as a coaxial cylinder, radial sector walls, or egg crate type of walls. This subdivision results in smaller sloshing masses and larger natural frequencies of the propellant. Another possibility is the clustering of tanks of smaller diameter, but this has structural and weight disadvantages. Radial walls for subdivision of a container[10, 14, 15] exhibit a more favorable result; one method in a particular case consists of subdividing a circular cylindrical tank by radial walls into four equal sector tanks. The natural frequencies of the propellant in the container increase, and the slosh masses decrease and are distributed to various modes, which is favorable to vehicle stability.

The theory of the response of a liquid in a circular-cylindrical quarter tank is presented for free and forced oscillations of the tank walls. Since the exact solution of this problem is too complicated to obtain the values of the natural frequencies and sloshing masses of the vibration modes, the liquid is treated as inviscid. The assumption of an inviscid liquid is justified, since the damping due to the friction of the tank walls is usually of very small magnitude. In a cylindrical container, the lower part of the liquid performs the forced motion like a rigid body, and only the propellant in the immediate vicinity of the free fluid surface oscillates by itself. For stability investigations of space vehicles, the motion perpendicular to the trajectory is of main importance. For this reason, our investigations are restricted to these motions. Free and forced vibrations in form of translatory, pitching, and roll excitation of the container will be treated. In these motions, the boundary conditions will be linearized. Besides this inherent simplification, this approach has the advantage that solutions of different excitations can be superimposed. The fact that the liquid is considered irrotational permits the representation of the velocity vector of the fluid as gradient of the velocity potential Φ.

Since the fluid is incompressible, the velocity potential must be a solution of the Laplace equation, which does not contain the time explicitly. This means that the flow pattern in the tank is at any time determined only by the boundary conditions holding at that time. Therefore, the analysis consists of solving the Laplace equation for various time- and space-dependent boundary conditions, which are, as well as the free fluid surface condition, linearized.

2. Basic Equations

An exact solution of the problem of liquid oscillations with a free fluid surface in a cylindrical container with circular-quarter cross section (Fig. 1) presents great difficulties. To obtain the most important features of such a liquid system, such as natural frequencies of the liquid and its response to various excitations, assume inviscid, irrotational, and incompressible flow. This means that the velocity potential Φ satisfies the Laplace equation

$$\nabla^2\Phi = 0 \tag{2.1}$$

The introduction of the velocity potential Φ has the advantage that all mechanically interesting values (flow velocities and pressures) can be derived from *one* single function. The velocity distribution is obtained by differentiation with respect to the spatial coordinates, and the pressure p from the instationary Bernoulli equation:

$$(\partial\Phi/\partial t) + \tfrac{1}{2}v^2 + (p/\bar{\rho}) + gz = 0$$

The complete solution of Eq. (2.1) must satisfy the following boundary conditions. At the tank walls, the normal velocity of the liquid and the wall must be equal. The free surface condition is obtained from the linearized kinematic $(\partial z/\partial t = \partial\Phi/\partial z)$ and the dynamic $(p = 0)$ condition.

3. Free and Forced Fluid Oscillations

To obtain the mode shapes and natural frequencies of the liquid, which are needed for the series expansion of the forced liquid oscillation, start with the free fluid oscillations.

The free oscillations of a liquid in a cylindrical container of radius a, circular-quarter cross section, and a free liquid surface can be obtained by solving the Laplace equation (2.1) with the linearized boundary conditions:

$$\partial\phi/\partial z = 0 \quad \text{at the container bottom } z = -h \tag{3.1a}$$

$$\partial\phi/\partial r = 0 \quad \text{at the circular wall } r = a \tag{3.1b}$$

$$(1/r)(\partial\phi/\partial\varphi) = 0 \quad \text{at the sector walls } \varphi = 0, \pi/2 \tag{3.1c}$$

$$(\partial^2\phi/\partial t^2) + g(\partial\phi/\partial z) = 0 \quad \text{at the free liquid surface } z = 0 \tag{3.1d}$$

The velocity potential becomes [considering the wall conditions (3.1a–3.1c) only]

$$\phi(r,\varphi,z,t) = \sum_m\sum_n A_{mn}e^{i\omega_{mn}t}\cos 2m\varphi \times \frac{\cosh\{\epsilon_{mn}[(z/a) + (h/a)]\}}{\cosh[\epsilon_{mn}(h/a)]} J_{2m}\left(\epsilon_{mn}\frac{r}{a}\right) \tag{3.2}$$

The values ϵ_{mn} are the positive roots of $J'_{2m}(\epsilon) = 0$. The equation for the frequencies of the liquid is obtained from the surface condition (3.1d):

$$\omega_{mn}^2 \equiv \omega^2 = \frac{g}{a}\epsilon_{mn}\tanh\left(\epsilon_{mn}\frac{h}{a}\right) \qquad m, n, = 0, 1, 2, \ldots \tag{3.3}$$

From this, one recognizes that the natural frequencies of the propellant are proportional to the root of the longitudinal acceleration and decrease as the inverse root of the container diameter. Therefore, for large containers, the natural frequencies of the liquid are small. For a certain tank filling, the natural frequencies of the liquid do not change any more with the liquid height because of $\tanh[\epsilon_{mn}(h/a)] \approx 1$ (see Table 1). The square of the natural frequencies is then $\omega^2 \approx g\epsilon_{mn}/a$. This expresses that the ratio $\omega/(g/a)^{1/2}$

Table 1 Root of $J_{2m}'(\epsilon_{mn}) = 0$

$n \backslash m$	0	1	2	3	4	5	6	7	8	9
0	3.832	3.054	5.318	8.105	9.648	11.716	13.821	15.917	18.104	20.189
1	7.016	6.706	9.282	11.735	14.115	16.448	18.745	21.015	23.264	25.495
2	10.173	9.969	12.682	15.268	17.774	20.223	22.629	25.002	27.347	29.670
3	13.324	13.370	15.964	18.637	21.229	23.761	26.246	28.694	31.112	33.504
4	16.471	16.348	19.196	21.932	24.587	27.182	29.729	32.237	34.712	37.160
5	19.616	19.513	22.401	25.184	27.889	30.535	33.131	35.689	38.212	40.707
6	22.760	22.672	25.590	28.410	31.155	33.842	36.481	39.079	41.643	44.178
7	25.904	25.826	28.768	31.618	34.397	37.118	39.792	42.426	45.052	47.595
8	29.047	28.978	31.939	34.813	37.620	40.371	43.075	45.740	48.371	50.971
9	32.189	32.127	35.104	38.000	40.830	43.607	46.338	49.030	51.687	54.315

changes its value only for small magnitudes of $h/a < 1$. At higher modes, this value is essentially constant $(\epsilon_{mn}^{1/2})$ except for very small values h/a (Fig. 2).

3.1 Forced Oscillations

For stability investigations of a space vehicle, the forces and moments exerted by the liquid propellant have to be known for an oscillation of the space vehicle about its normal trajectory. These motions are translatory oscillations perpendicular to the flight path and rotational oscillations about the lateral and the longitudinal axes. For this reason the various cases of a container performing forced oscillations are investigated. Since the liquid in the lower part of the container performs the same motion as the tank, it is useful to express the potential Φ as a sum of the rigid body potential ϕ_0 and a disturbance potential ϕ:

$$\Phi = \phi_0 + \phi \tag{3.4}$$

which is caused by the free surface motion.

3.1.1 Translational excitation

The special case that describes the response of the liquid in a cylindrical container of circular-quarter cross section due to translational excitation of the container is obtained from the Laplace equation (2.1) with the boundary conditions

$$\partial\Phi/\partial r = i\Omega e^{i\Omega t}x_0 \cos\varphi \quad \text{at the cylinder wall } r = a \tag{3.5a}$$

$$\partial\Phi/\partial z = 0 \qquad \text{at the tank bottom } z = -h \tag{3.5b}$$

$$\partial\Phi/r\partial\varphi = 0 \qquad \text{at the sector wall} = 0 \tag{3.5c}$$

$$\partial\Phi/r\partial\varphi = -i\Omega x_0 e^{i\Omega t} \quad \text{at the sector wall} = \pi/2 \tag{3.5d}$$

$$(\partial^2\Phi/\partial t^2) + g(\partial\Phi/\partial z) = 0 \quad \text{at the free liquid surface } z = 0 \tag{3.5e}$$

By separating the container motion from the potential Φ, the wall boundary conditions of the disturbance potential can be made homogeneous. With

$$\Phi = [i\Omega r x_0 \cos\varphi + \phi]e^{i\Omega t} \tag{3.6}$$

the boundary conditions are then

$$\partial\phi/\partial r = 0 \qquad \text{for } r = a$$

$$\partial\phi/\partial z = 0 \qquad \text{for } z = -h$$

$$\partial\phi/r\partial\varphi = 0 \qquad \text{for } \varphi = 0, \pi/2$$

$$g(\partial\phi/\partial z) - \Omega^2\phi = i\Omega^3 r x_0 \cos\varphi \qquad \text{for z} = 0$$

The disturbance potential, which satisfies the Laplace equation, has the same form as (2.4). For the sake of a clearer representation, omitting the summation signs and the indices m and n, and introducing the abbreviations $\zeta = \epsilon_{mn}(z/a)$, $\rho = \epsilon_{mn}(r/a)$, $\kappa = \epsilon_{mn}(h/a)$, and $J(\rho) = J_{2m}[\epsilon_{mn}(r/a)]$, the disturbance potential is

$$\phi(r,\varphi,z) = AJ(\rho)\cos 2m\varphi \frac{\cosh(\zeta + \kappa)}{\cosh\kappa} \tag{3.7}$$

From the free surface condition, one obtains the unknown constants A_{mn} if one expands $\cos\varphi$ in a Fourier series and the function r in a Bessel series.[16] With $\eta = \Omega/\omega$ as the ratio of exciting to natural frequency, the velocity potential Φ for translatory excitation of the container in the x direction is

$$\Phi(r,\varphi,z,t) = i\Omega x_0 e^{i\Omega t}\left\{r\cos\varphi + \frac{a_m b_{mn} J(\rho)\eta^2 \cosh(\kappa + \zeta)\cos 2m\varphi}{(1 - \eta^2)\cosh\kappa}\right\} \tag{3.8}$$

The first term (rigid body potential) satisfies the boundary conditions at the container walls, whereas the second part (disturbance potential) vanishes there. The condition of the free surface is satisfied with both terms. The free surface displacement measured from the undisturbed position is

$$\bar{z} = -\frac{1}{g}\left(\frac{\partial\Phi}{\partial t}\right)_{z=0} = x_0 e^{i\Omega t}\frac{\Omega^2}{g/a} \times \left[\frac{r}{a}\cos\varphi + \frac{a_m b_{mn}\eta^2 J(\rho)\cos 2m\varphi}{a(1 - \eta^2)}\right] \tag{3.9}$$

The pressure distribution in the liquid is at a depth $(-z)$:

$$p = -\bar{\rho}\frac{\partial\Phi}{\partial t} - \bar{\rho}gz = \bar{\rho}\Omega^2 x_0 e^{i\Omega t} \times \left[r\cos\varphi + \frac{a_m b_{mn}\eta^2 \cosh(\zeta + \kappa)J(\rho)}{(1 - \eta^2)\cosh\kappa}\cos 2m\varphi\right] - \bar{\rho}gz \tag{3.10}$$

At the tank wall, $r = a$, the pressure is obtained from (3.10) by setting $J(\rho)$ equal to $J_{2m}(\epsilon_{mn})$. At the sector walls $\varphi = 0$ and $\varphi = \pi/2$, the cosine $(\cos 2m\varphi)$ assumes value one and $(-1)^m$. The pressure distribution at the tank

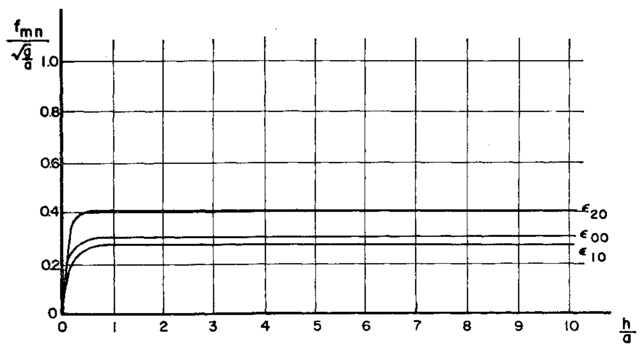

Fig. 2 Eigenfrequency parameter.

bottom is obtained by setting $\zeta = -\kappa(z = -h)$. From the pressure distribution, one obtains by integration the fluid forces and moments. The resulting force in the x direction is

$$F_x = a \int_0^{\pi/2} \int_{-h}^{0} p_{\text{wall}} \cos\varphi d\varphi dz - \int_0^a \int_{-h}^{0} p_{\varphi=\pi/2}\, dr\, dz \tag{3.11}$$

The first integral represents the contribution of the pressure

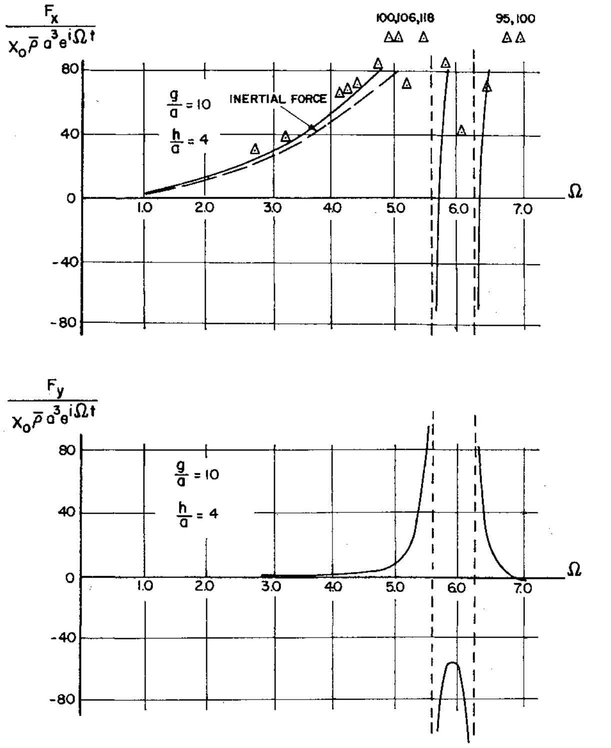

Fig. 3 Magnification factor of liquid forces for excitation in x direction (measured values[17]).

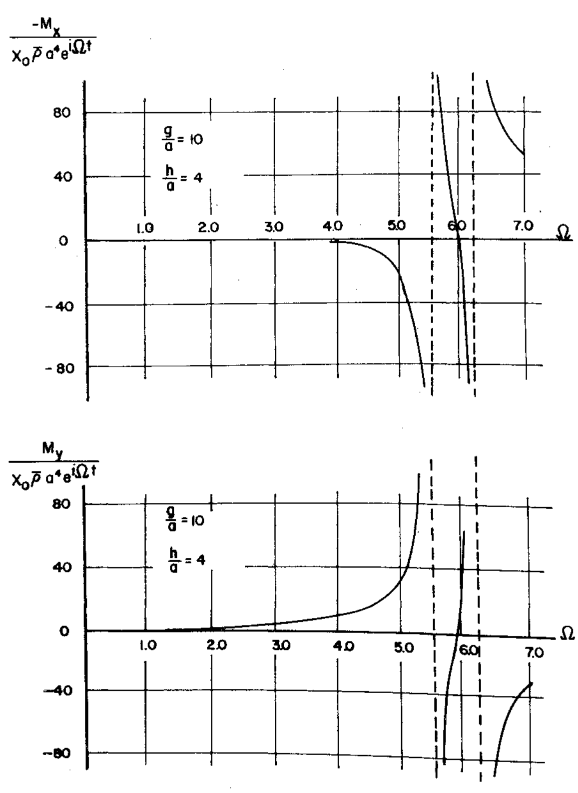

Fig. 4 Magnification factor for liquid moments for excitation in x direction.

at the circular wall, whereas the second integral stems from the sector wall at $\varphi = \pi/2$. The fluid force is (Fig. 3)

$$F_x = m\Omega^2 x_0 e^{i\Omega t}\left[1 + \frac{4\eta^2}{\pi a}\frac{(-1)^{m+1}a_m b_{mn}\tanh\kappa}{(1-\eta^2)\kappa} \times \left\{\frac{J_{2m}(\epsilon_{mn})}{(4m^2-1)} + L_0(\epsilon_{mn})\right\}\right] \tag{3.12}$$

where

$$L_0(\epsilon_{mn}) = \frac{2}{\epsilon_{mn}}\sum_{\mu=0}^{\infty} J_{2m+2\mu+1}(\epsilon_{mn})$$

The first term in (3.12) represents the inertial force, that is, the

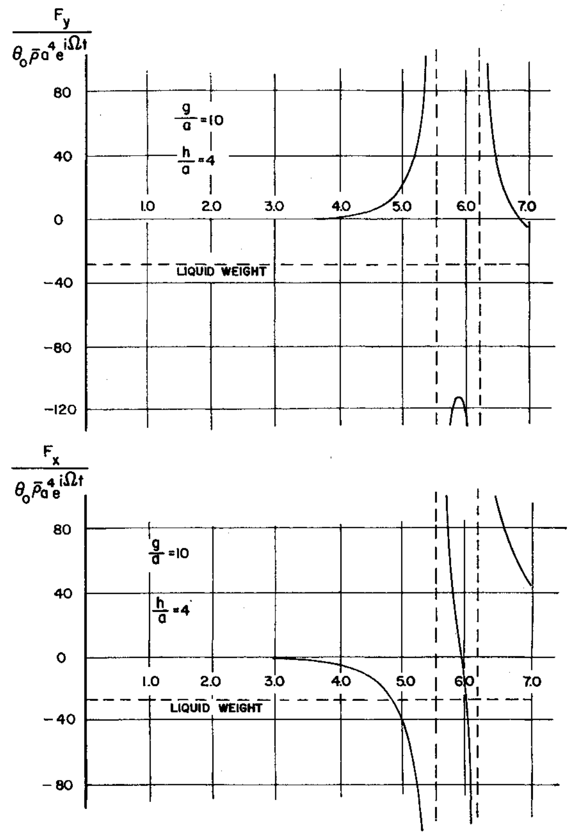

Fig. 5 Magnification factor for liquid forces for excitation about y axis.

force for a solidified liquid. The liquid force in y direction is

$$F_y = m\Omega^2 x_0\, e^{i\Omega t}\frac{4a_m b_{mn}\eta^2 \tanh\kappa}{\pi a\kappa(1-\eta^2)} \times \left\{\frac{J_{2m}(\epsilon_{mn})}{(4m^2-1)} + L_0(\epsilon_{mn})\right\} \tag{3.13}$$

The moment M of the liquid with respect to the point $(0, 0, -h/2)$ is given by

$$M_y = a\int_0^{\pi/2}\int_{-h}^{0} p_{\text{wall}}\left(\frac{h}{2}+z\right)\cos\varphi d\varphi dt - \int_0^a\int_{-h}^{0} p_{\varphi=\pi/2}\left(\frac{h}{2}+z\right) dr dz + \int_0^{\pi/2}\int_0^a p_{\text{bottom}}\, r^2\cos\varphi d\varphi dr \tag{3.14}$$

The first integral again stems from the pressure distribution at the circular wall, whereas the second integral represents the contribution of the pressure at the sector wall. The third part of the formula is due to the pressure at the tank

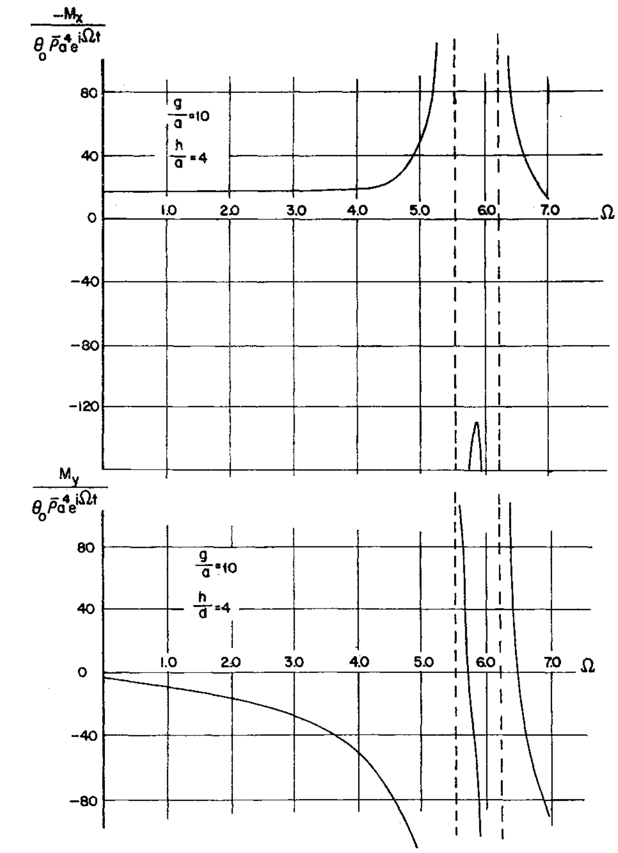

Fig. 6 Magnification factor for liquid moments for excitation about *y* axis

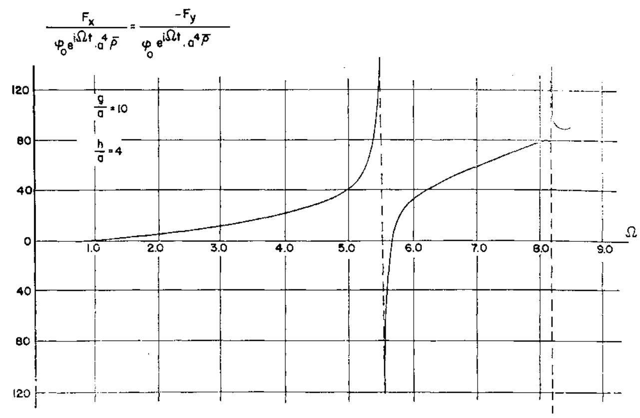

Fig. 7 Magnification factor of liquid forces for roll excitation.

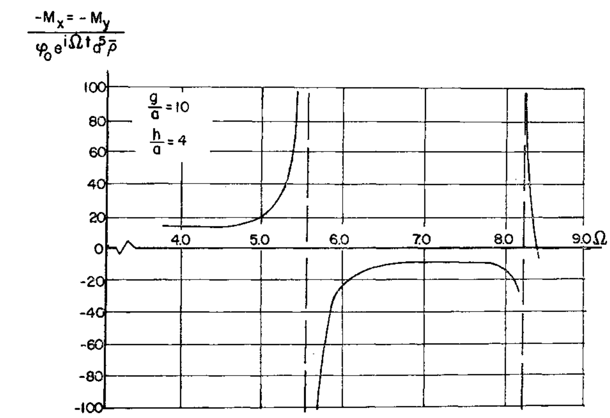

Fig. 8 Magnification factor of liquid moments $M_x = M_y$ for roll excitation.

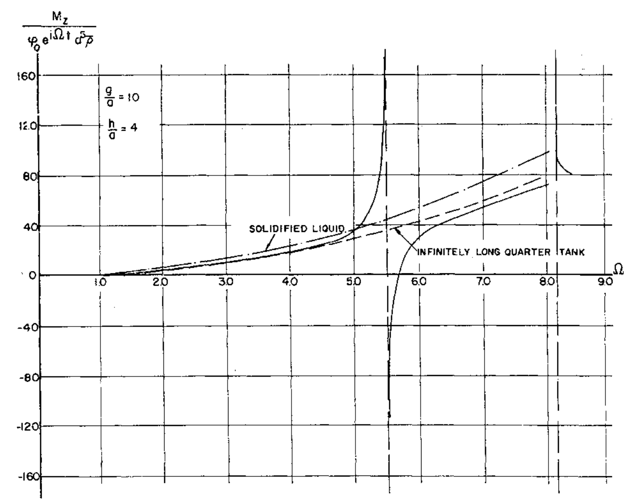

Fig. 9 Magnification factor of liquid moment M_z for roll excitation.

bottom. After integration, the moment is (Fig. 4)

$$M_y = m\Omega^2 a x_0 e^{i\Omega t}\left[\frac{1}{4h/a} + \frac{2}{\pi}\frac{a_m b_{mn}\eta^2(-1)^{m+1}}{a\epsilon_{mn}(1-\eta^2)} \times \left\{\left[\frac{J_{2m}(\epsilon_{mn})}{(4m^2-1)} + L_0(\epsilon_{mn})\right]\left[\tanh\kappa + \frac{2}{\kappa}\left(\frac{1}{\cosh\kappa} - 1\right)\right] + \frac{2\epsilon_{mn}{}^2 L_2(\epsilon_{mn})}{(4m^2-1)\kappa\cosh\kappa}\right\}\right] + \frac{4mga}{3\pi} \qquad (3.15)$$

Since the reference axis does not pass through the center of gravity, the last term of Eq. (3.15) exhibits the static moment of the liquid. The moment M_x is obtained in a similar way and is

$$M_x = m\Omega^2 a x_0 e^{i\Omega t}\left[\frac{1}{\pi h/a} - \frac{2}{\pi}\frac{a_m b_{mn}\eta^2}{a\epsilon_{mn}(1-\eta)^2} \times \left\{\left[\frac{J_{2m}(\epsilon_{mn})}{(4m^2-1)} + L_0(\epsilon_{mn})\right]\left[\tanh\kappa + \frac{2}{\kappa}\left(\frac{1}{\cosh\kappa} - 1\right)\right] + \frac{2\epsilon^2_{mn} L_2(\epsilon_{mn})}{(4m^2-1)\kappa\cosh\kappa}\right\}\right] - \frac{4mga}{3\pi} \qquad (3.16)$$

where

$$L_2(\epsilon_{mn}) = \frac{2(4m^2-1)}{\epsilon_{mn}}\sum_{\mu=0}^{\infty}\frac{J_{2m+2\mu+1}(\epsilon_{mn})}{(2m+2\mu-1)(2m+2\mu+3)}$$

The response of the liquid due to rotational excitation such as pitching or roll oscillations can be obtained in a similar way. The results are presented in Figs. 5–9.

4. Effective Moment of Inertia of Liquid

In the description of the liquid motion as a mechanical model, the effective moment of inertia of the liquid in a completely filled and closed container has to be known. It can be obtained by solving the Laplace equation with the same boundary conditions as in the *pitching* case, except that, for the free fluid surface condition, the boundary condition for the bottom is used with $z = +h/2$. The velocity potential is then

$$\Phi(r,\varphi,z,t) = i\Omega\Theta_0 e^{i\Omega t}\left[-rz\cos\varphi + \frac{2a\, a_m b_{mn}\sinh\zeta J(\rho)}{\epsilon_{mn}\cosh(\kappa/2)}\cos 2m\varphi\right]$$

The pressure distribution is obtained from

$$p = -\bar{\rho}\,\frac{\partial\Phi}{\partial t} + g\bar{\rho}\left[\frac{h}{2} - z + \Theta_0 e^{i\Omega t}\,(a - r\cos\varphi)\right] =$$

$$\bar{\rho}\Omega^2\Theta_0 e^{i\Omega t}\left[-rz\cos\varphi + \frac{2a\,a_m b_{mn}\sinh\zeta J(\rho)}{\epsilon_{mn}\cosh\kappa/2}\cos 2m\varphi\right] + \bar{\rho}g\left[\frac{h}{2} - z + \Theta_0 e^{i\Omega t}\,(a - r\cos\varphi)\right]$$

With the introduction of an additional integral due to the pressure distribution at the top of the container, the moment can be determined. The effective moment of inertia of the liquid is then

$$I_{y\,\mathrm{eff}} = ma^2\left\{\frac{1}{12}\left(\frac{h}{a}\right)^2 - \frac{1}{4} + \frac{2(-1)^m a_m b_{mn}}{a\pi\epsilon_{mn}{}^2}\times \left[\left(1 - \frac{2\tanh(\kappa/2)}{\kappa}\right)\left(\frac{J_{2m}(\epsilon_{mn})}{(4m^2-1)} + L_0(\epsilon_{mn})\right)\right] - \frac{2\epsilon_{mn}{}^2 L_2(\epsilon_{mn})\tanh\kappa/2}{(4m^2-1)\kappa}\right\}$$

The moment of the rigid body is

$$I_{y\,\mathrm{rigid}} = ma^2\left\{\tfrac{1}{12}(h/a)^2 + \tfrac{1}{4}\right\}$$

For roll excitation, the effective moment of inertia of the liquid can be obtained and is

$$I_{z\,\mathrm{eff}} = \frac{ma^2}{2}\left[1 - \left(\frac{4}{3} - \frac{16}{\pi^2}\ln 2\right)\right] = \frac{ma^2}{2}\cdot 0.7903$$

where the first term represents the moment of inertia of the rigid body.

5. Comparison with Experimental Results

Comparison of the theoretical natural frequencies with those of the measured values shows, as was already indicated in Ref. 17, too high values. This comparison indicates that the natural frequencies of the liquid in compartmented tanks depend strongly upon the magnitude of the excitation amplitude. They approach the theoretical values only when the excitation is infinitesimal. It was found that the experimental natural frequencies deviate from the theoretical for an excitation amplitude of $x_0 = 0.0016\,a$ by about 8%. Perforation of the sector walls can maintain, by proper choice of the hole size, the mechanical features of the quarter tank with some weight savings.

Since propellant sloshing occurs only in the upper portion of the liquid and since the sloshing masses do not vary considerably for values of $h/a > 1$, the experimental values[17] compare favorably with the theoretical results of the present paper (see Fig. 3).

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